ON A CERTAIN CLASS OF INCORRECTLY STATED PROBLEMS IN ANALYTICAL THEORY OF THERMAL CONDUCTIVITY AND IN THE THEORY OF INDIRECT MEASUREMENTS

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The variational problem of stationary thermal conductivity in an inhomogeneous solid is formulated. It is assumed that the boundary conditions on the boundary of the body are unknown. In order to obtain a unique and stable solution one requires measurement of the temperature at one point and correct selection of the regularization parameter.

As is well known, an indirect measurement is defined as a measurement in which the value of the desired quantity is determined by calculation on the basis of direct measurements of other quantities which are associated with the measured quantity by a known dependence. Under these conditions the procedure for calculating the desired quantity may be fairly complicated.

If the relationship between the desired and measured quantities is known with an accuracy of up to a certain functional equation reflecting, for example, the condition of conservation of energy, momentum or mass, the superposition principle or other physical laws, then the notion of indirect measurement acquires a meaning that is different from one usually accepted in metrology.

In engineering the necessity frequently arises of measuring a certain physical parameter where one cannot, for example, place a meter in view of structural, technological, or operational peculiarities (i.e., factors based on an especially practical property).

Let it be required to measure the temperature on the surface of a body or to determine the heat flux through its boundary for heating by an external source. For various reasons the possibility of placing the sensing elements of a transducer on the surface of the body or of performing measurements by the contactless method is excluded.

If the temperature depends on one coordinate, then for the simplest body (half-space, plate, cylinder, sphere) the problem indicated is solved as an inverse problem or as a problem in reconstruction (analytical continuation), depending on the number of points inside the body at which measurement is performed [1, 2, 3]. In other words, finding the temperature at some point of the body from a measurement at a different point of the body is essentially an indirect measurement, but not in the sense of the definition generally accepted in metrology.

As will be evident from the subsequent expanded statement of the problem, in terms of the mathematical procedure this type of measurement is close to the operational treatment of indirect measurement in quantum mechanics.

However, the mathematical operation of reconstructing values of a function or the function itself from one or several measurements may be incorrect, since one of the three conditions for a correctly stated problem is not always fulfilled, notwithstanding the existence of physical uniqueness of the situation.

Regularization methods proposed by Tikhonov, Morozov, Turchin, and associates [4], and by a series of many mathematicians eliminate the incorrectness of the original problem in various ways.

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Let us go over to formulation of the following important practical problem.

In a continuous body but one which is inhomogeneous in its thermophysical properties† it is required to determine the temperature at some points $X^* = (x^*, y^*, z^*)$ where direct measurement is impossible, but the temperature at one or several other points $X_0, X_1, \ldots,$ may be monitored.

Let us consider the case of stationary thermal conductivity.

Let a body consist of a union of domains $D_1, D_2, \ldots, D_n : D = \bigcup_{i=1}^{n} D_i$ which are homogeneous in their properties and have different thermophysical properties. Domains containing liquids or gas or evacuated cavities may likewise serve as D_i. Under these conditions it is assumed that there is no convective motion in a gas or liquid.

Inside the domains D_i the presence of energy sources or sinks is not excluded by definition.

Heat exchange with the ambient medium may go on according to various laws on the outside surface of the body. However, these laws are assumed to be unknown. In this case it is required to determine the temperature at the point X^*, as well as the boundary conditions, from one measurement at the point X_0 (or at several points). The point X_0 may lie on the boundary of the body; the function of a point may be performed by a certain surface or volume whose temperature is monitored.

In the body considered, which fills a three-dimensional domain D having a boundary Ω, we introduce a Cartesian coordinate system with rectangular, spherical, or cylindrical symmetry; this will be determined by the delineation or properties of the boundary Ω that closes the union of domains $D_1, D_2, \ldots, D_n$.

Let us write the Onsager–Glansdorff–Prigogine maximum principle [5, 6, 7] for the production of entropy in the stationary state of a body on the assumption that the boundary conditions are fixed in space and time. Based on the additiveness property of entropy, entropy production, and entropy flux as quantities of an extensive character, the maximum principle for the problem stated will have the form

$$J = -\int\limits_{V=\bigcup_{i=1}^{n} D_i} \left[\frac{\lambda(D_i)}{2}(\nabla T)^2 + T\cdot w\right] dv + \int\limits_{\Omega=\bigcup_{l=1}^{m} D_l(\Gamma)} \lambda(D_l(\Gamma))\, T\cdot \frac{\partial T^0}{\partial n}\cdot d\Omega = \max, \tag{1}$$

where T(X) is the temperature at an arbitrary point of the body; w(X) is the power of internal heat sources or sinks.

The second term in the functional (1) is the steady-state entropy flux through the boundary Ω.

Since the operator which stipulates the distribution of the temperature field in a homogeneous body having the boundary Ω is known, its eigenfunctions $\Psi(X) = \{\psi_i(x), \psi_j(y), \psi_k(z)\}$ are known. In other words, the eigenfunctions $\{\psi\}_{i,j,k}$ are stipulated according to a similar (homogeneous linear) operator [8].

The eigennumbers $\{\mu_i, \mu_j, \mu_z\} = \{\mu\}_{i,j,k}$ corresponding to the eigenfunctions $\{\psi\}_{i,j,k}$ may likewise be stipulated on the assumption that the body is homogeneous and consists of the material of the subdomain D_r in which the origin is situated, while the boundary Ω is impenetrable to thermal flux.

The latter proposition will be considered somewhat later.

Thus the temperature field may be represented in the form of the series

$$\tilde{T}(X) = \sum_{ijk} a_{ijk}\tilde{\psi}_i(\mu_i, x)\,\tilde{\psi}_j(\mu_j, y)\,\tilde{\psi}_k(\mu_k, z), \tag{2}$$

where $\tilde{\psi}$ is the approximate value of the eigenfunction for the original problem ($\{\mu\}_{i,j,k}$ are inexactly stipulated).

The temperature gradient ∇T undergoes violation of continuity when the transition is made through the interfaces between the domains D_i. In the stationary state its jumps will be determined by the ratio between the coefficients of thermal conductivity. Let us write the gradient of T(X) after stipulating a stepwise change of its projections:

† In engineering practice cases in which the physical coefficients vary stepwise are most frequently encountered.

$$\nabla \tilde{T}(X) = \left\{ \frac{\lambda(D_i)}{\lambda(D_r)} \left[\sum_{ijk} a'_{ijk} \tilde{\psi}_{i,i}\, \psi_j \tilde{\psi}_k \right]; \right.$$
$$\frac{\lambda(D_i)}{\lambda(D_r)} \left[\sum_{ijk} a''_{ijk} \tilde{\psi}_i \psi_{j,j} \tilde{\psi}_k \right]; \tag{3}$$
$$\left. \frac{\lambda(D_i)}{\lambda(D_r)} \left[\sum_{ijk} a'''_{ijk} \psi_i\, \psi_j\, \psi_{k,k} \right] \right\},$$

where the notation

$$\psi_{i,i} = \frac{\partial \tilde{\psi}_i}{\partial x},\ \tilde{\psi}_{j,j} = \frac{\partial \tilde{\psi}_j}{\partial y},\ \tilde{\psi}_{k,k} = \frac{\partial \tilde{\psi}_k}{\partial z} \tag{4}$$

has been used.

The stepwise variations of the gradients correspondingly define a stepwise variation of the magnitude of the entropy production for the transition from domain to domain:

$$J = -\left[\int\limits_{V = \bigcup\limits_{i=1}^{n} D_i} \frac{\lambda(D_i)}{2} \cdot \frac{\lambda^2(D_i)}{\lambda^2(D_r)} \left[\tilde{T}^2_{,i} + \tilde{T}^2_{,j} + \tilde{T}^2_{,k} \right] dV \right.$$
$$\left. + \int\limits_{V = \bigcup\limits_{i=1}^{n} D_i} \tilde{T} \cdot w(X) dV \right] + \int\limits_{\Omega = \bigcup\limits_{l=1}^{m} D_l(\Gamma)} \lambda(D_l(\Gamma)) \tilde{T} \frac{\partial \tilde{T}^0}{\partial n} \cdot d\Omega = \max. \tag{5}$$

Holding to the terminology introduced by Tikhonov [9], we shall call the variational problem of the search for an extremum of a continuous convex functional to be correctly stated in a generalized manner if the following conditions are satisfied:

1) the set of solutions causing the original function to have a minimum or maximum is not empty;
2) the arbitrary minimizing sequence converges in the norm to the exact solution in the corresponding metric.

The first condition is fulfilled on the basis of the physical essence of the stated problem. For the time being nothing may be said concerning the second condition.

In [3, 9–11] it has been shown that in variational problems involving the extremum of a continuous convex functional (optimal control problems) the minimizing sequences in general do not converge strongly or even weakly to an exact solution. Tikhonov [9] proposed a regularization method ensuring uniform convergence of the minimizing sequence in the presence of a fairly smooth solution of the original problem in order to solve incorrectly stated variational problems. Later Budak et al. [10] proposed regularization methods with the formulation of minimizing sequences having convergence in the norm on continuity intervals (i.e., for the case of piecewise-continuous extremals of the functional).

Under these conditions the algorithm for formulating the minimizing sequences mentioned, which are based on conditional-gradient methods, the gradient-projection method, etc. [10], are complicated to realize and as yet are not extensively used in applied engineering problems.

As Levitin and Polyak showed [10, 11], in the case of a variational problem with a simple constraint of the linear rigorous or nonrigorous inequality type, an effective device for achieving a solution remains the Ritz–Rayleigh method.

We shall discuss a constraint for the problem we have stated a little further on.

Returning to the functional (4), (1) we note that its properties involving continuity, convexity, and boundedness from below have recently been proved in [12]. This fact simplifies the subsequent consideration of the problem somewhat, since, having in mind [3] and [10], there is no need for us to prove strong convergence of the minimizing sequence for an exact solution for the corresponding regularized functionals. The functional (5), for which the entropy flux through the surface is equal to zero (i.e., for the case of an isolated body), has analogous properties.

Let us dwell in somewhat greater detail on this feature.

The fact that in the statement of the problem the boundary conditions are unknown necessarily leads to the idea of placing the external entropy flux equal to zero and introducing a different uniqueness requirement instead of the boundary conditions, namely

$$\tilde{T}(X_0) = T_0, \tag{6}$$

where T_0 is the measured temperature.

Condition (6) simultaneously plays the role of a boundary requirement during the minimization of the functional.

Thus, we have the following variational problem:

$$J(\tilde{T}) = \int\limits_{V=\bigcup\limits_{i=1}^{n} D_i} \left[\frac{\lambda(D_r)}{2} \frac{\lambda^2(D_i)}{\lambda^2(D_r)} [\tilde{T}^2_{i,i} + \tilde{T}^2_{j,j} + \tilde{T}^2_{k,k}] + w(X)\tilde{T} \right] dV = \min, \tag{7}$$

$$\tilde{T}(X_0) = T_0. \tag{8}$$

The functional (7) is the expression for the Onsager principle of minimum dissipative energy for the case of an inhomogeneous body, or, in the alternative and preferable formulation of Prigogine, the principle of least entropy production for the stationary state.

Let us now undertake the "solution" of the system (7), (8).

Let us identify the space of functions on which we shall seek the extremal of the functional with Hilbert space* which shall be realized further on as L_2.

Let us write the regularized functional (1) in the form

$$J_{\alpha_m}(T) = J(T) + \alpha_m \|T\|^2_{L_2}, \quad 0 < \alpha_m < A, \; \alpha_m \to 0. \quad m \to \infty. \tag{9}$$

Here and further on we shall hold to the notation used in [10].

Let us consider the sequence of regularized approximating functionals

$$J_{\varepsilon_m \alpha_m} = J(T) + \alpha_m \|T\|^2_{L_2} + \Theta_m(T), \tag{10}$$

where $|\Theta_m(T)| < \varepsilon_m$; ε_m is the approximation error which is uniform on the entire closed convex set of solutions τ, $T \in \tau$. Under these conditions $\varepsilon_m \to 0$, $m \to \infty$.

Having (2) in mind, the functional (10) may be written thus:

$$J_{\varepsilon_m \alpha_m} = J(\tilde{T}) + \alpha_m \|\tilde{T}\|^2_{L_2}. \tag{11}$$

The functionals (9)-(11) are bounded from below, continuous, and strongly convex, which is a necessary condition for formulating a strongly convergent minimizing sequence [3].

The determination of the sequence α_m is the key feature from the point of view of the practical solution of the variational problem. Therefore let us dwell in greater detail on the procedures for choosing α_m.

The choice of the sequence α_m is determined by the three following factors:

1) the incorrectness of the variational problem (absence of strong convergence of the minimizing sequence);

2) instability of the solution because of errors in stipulating the measured temperature T_0;

*In general, if we have in mind weak convergence to the exact solution it follows that the functional space is not Hilbertian but is close to the concept of an energy space (but not in the sense of an active state, after Friedrichs, but rather in the sense of a dissipative state, i.e., a space of scattered energy generated by a conservative dissipative operator [13]).

3) inaccuracy in stipulating the eigennumbers in the expansion (2).

In the paper by Tikhonov [9] items 1 and 2 listed above cannot be separated, and it is said that if the sequence α_m exists it follows that besides uniform convergence of the approximate solutions to the exact solution for the variational problem, stability of the solution relative to variation of the errors in the original conditions is likewise assured.

Let us now introduce what is evidently an assumption which is not very burdensome: we identify item 3 with item 1. This allows methods which have already been developed to be used in formulating the minimizing sequence α_m. So far the simplest method is that of Morozov [3].

All necessary proofs concerning the properties of the principal solutions of the regularized problem have been given in [3]. Therefore it is required merely to relate the procedures for choosing α_m to the Ritz–Rayleigh method of formulating a minimizing sequence.

In the expansion (2) we limit the analysis to s terms and assume $\tilde{T}_s = \tilde{T}^0(\alpha = 0)$ to be a "trial" solution. Let us choose certain positive numbers $\alpha > 0$. We calculate the values of the functionals (with allowance for the constraint (8)) $J(\tilde{T}^0)$ and $J_\alpha(\tilde{T}^0)$. We determine the difference

$$\|J_\alpha(\tilde{T}^0) - J(\tilde{T}^0)\|_{L_2} = \delta. \tag{12}$$

Let us designate the lower bound $\underline{\alpha} > \delta$. We now take a certain $\alpha_{\max} > \underline{\alpha}$ but one which is such that $\tilde{T}_{\alpha_{\max}} \sim \tilde{T}^0$. Finally, we choose the numerical value of τ, $0 < \tau < 1$ and formulate the sequence

$$\alpha_m = \tau^m(\alpha_{\max} - \underline{\alpha}), \quad m = 0, 1, 2, \ldots \tag{13}$$

We shall add terms to the expansion (2), thus increasing the number of terms of the series and simultaneously multiplying out $\|\tilde{T}\|^2_{L_2}$ by the corresponding value of α_m. From the sequence of systems obtained

$$J_{\varepsilon_m \alpha_m} = J(\tilde{T}) + \alpha_m \|\tilde{T}\|^2_{L_2}, \tag{14}$$

$$\tilde{T}(X_0) = T_0 \tag{15}$$

we determine the coefficients a_{ijk}. As the criterion for estimating the closeness of the approximate solution $\tilde{T}_{\alpha_m}$ to the true solution we use the norm

$$\|\tilde{T}_{\alpha_m} - \tilde{T}_{\alpha_{m-1}}\|_{L_2} < \Delta, \tag{16}$$

where Δ is the stipulated error.

In order to designate the "trial" solution $\tilde{T}^0$ it is evidently necessary to take at least 4 to 5 eigennumbers and eigenfunctions for each coordinate.

An experimental check of the method which has been developed above on mathematical models constitutes a subject for subsequent investigation.

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